

SIMPLE GRADED RINGS, NON-ASSOCIATIVE CROSSED PRODUCTS AND CAYLEY-DICKSON DOUBLINGS

PATRIK NYSTEDT

University West, Department of Engineering Science, SE-46186 Trollhättan, Sweden

JOHAN ÖINERT

*Blekinge Institute of Technology, Department of Mathematics and Natural Sciences,
SE-37179 Karlskrona, Sweden*

ABSTRACT. We show that if a non-associative unital ring is graded by a hypercentral group, then the ring is simple if and only if it is graded simple and the center of the ring is a field. Thereby, we extend a result by Jespers from the associative case to the non-associative situation. By applying this result to non-associative crossed products, we obtain non-associative analogues of results by Bell, Jordan and Voskoglou. We also apply this result to Cayley-Dickson doublings, thereby obtaining a new proof of a classical result by McCrimmon.

1. INTRODUCTION

Throughout this article, let R be a (not necessarily associative) unital ring whose multiplicative identity is denoted by 1. Let G be a multiplicatively written group with identity element e . If there are additive subgroups R_g , for $g \in G$, of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$, for $g, h \in G$, then we shall say that R is *graded by G* (or *G -graded*). If in addition $R_g R_h = R_{gh}$, for $g, h \in G$, then R is said to be *strongly graded by G* (or *strongly G -graded*).

The investigation of associative graded rings has been carried out by many authors (see e.g. [11, 12] and the references therein). Since many ring constructions are special cases of graded rings, e.g. group rings, twisted group rings, (partial) skew group rings and crossed product algebras (by twisted partial actions), the theory of graded rings can be applied to the study of other less general constructions, giving new results for several constructions simultaneously, and unifying theorems obtained earlier.

E-mail addresses: patrik.nystedt@hv.se, johan.oinert@bth.se.

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An important problem in the investigation of graded rings is to explore how properties of R are connected to properties of subrings of R . In the associative case, many results of this sort are known for finiteness conditions, nil and radical properties, semisimplicity, semiprimeness and semiprimitivity (see e.g. [7, 8]).

In this article, we shall focus on the property of R being *simple*. An obvious *necessary* condition for simplicity of R is that the center of R , denoted by $Z(R)$, is a field. This condition is of course not in general a *sufficient* condition for simplicity of R . However, if it is combined with *graded simplicity*, then in the associative setting Jespers [5] has shown (see Theorem 1) that, for many groups, it is. To be more precise, recall that an ideal I , always assumed to be two-sided, of R is called *graded* if $I = \bigoplus_{g \in G} (I \cap R_g)$. The ring R is called *graded simple* if the only graded ideals of R are $\{0\}$ and R itself. The group G is called *hypercentral* if every non-trivial factor group of G has a non-trivial center. Note that all abelian groups and all nilpotent groups are hypercentral (see [18]).

Theorem 1 (Jespers [5]). *If an associative unital ring is graded by a hypercentral group, then the ring is simple if and only if it is graded simple and the center of the ring is a field.*

This result has far-reaching applications. Indeed, Jespers [5] has applied Theorem 1 to obtain necessary and sufficient criteria for simplicity of crossed product algebras, previously obtained by Bell [1] (see Theorem 2), and for skew group rings, previously obtained by Voskoglou [20] and Jordan [6] (see Theorem 3). Let T be an associative unital ring and let the multiplicative group of units of T be denoted by T^\times . Suppose that (T, G, σ, α) is a *crossed system*. Recall that this means that G is a group, and that $\sigma : G \rightarrow \text{Aut}(T)$ and $\alpha : G \times G \rightarrow T^\times$ are maps satisfying the following three conditions for any triple $g, h, s \in G$ and any $a \in T$:

- (i) $\sigma_g(\sigma_h(a)) = \alpha(g, h)\sigma_{gh}(a)\alpha(g, h)^{-1}$;
- (ii) $\alpha(g, h)\alpha(gh, s) = \sigma_g(\alpha(h, s))\alpha(g, hs)$;
- (iii) $\sigma_e = \text{id}_T$ and $\alpha(g, e) = \alpha(e, g) = 1$.

The corresponding *crossed product*, denoted by $T \rtimes_\sigma^\alpha G$, is the collection of finite sums $\sum_{g \in G} t_g u_g$, where $t_g \in T$, for $g \in G$, equipped with coordinate-wise addition and multiplication defined by the bi-additive extension of the relations $(a u_g)(b u_h) = a \sigma_g(b) \alpha(g, h) u_{gh}$, for $a, b \in T$ and $g, h \in G$. If we put $(T \rtimes_\sigma^\alpha G)_g = T u_g$, for $g \in G$, then this defines a strong G -gradation on the crossed product. Let T^G denote the *fixed subring* of T , i.e. the set of all $t \in T$ satisfying $\sigma_g(t) = t$, for $g \in G$. An ideal I of T is said to be *G -invariant* if for every $g \in G$, the inclusion $\sigma_g(I) \subseteq I$ holds. The ring T is said to be *G -simple* if the only G -invariant ideals of T are $\{0\}$ and T itself. It is easy to see that T is G -simple precisely when $T \rtimes_\sigma^\alpha G$ is graded simple (see Proposition 38). Let $Z(G)$ denote the center of G .

Theorem 2 (Bell [1]). *If $T \rtimes_\sigma^\alpha G$ is a crossed product, where G is a torsion-free hypercentral group and T is G -simple, then the following three conditions are equivalent: (i) $T \rtimes_\sigma^\alpha G$ is simple; (ii) $Z(T \rtimes_\sigma^\alpha G) = Z(T)^G$; (iii) there do not exist $u \in T^\times$ and a non-identity $g \in Z(G)$ such that for every $h \in G$ and every $t \in T$, the relations $\sigma_h(u) = \alpha(g, h)^{-1} \alpha(h, g) u$ and $\sigma_g(t) = u t u^{-1}$ hold.*

If a crossed product $T \rtimes_{\sigma}^{\alpha} G$ satisfies $\alpha(g, h) = 1$, for all $g, h \in G$, then it is called a *skew group ring* and is then denoted by $T \rtimes_{\sigma} G$; in this case $\sigma : G \rightarrow \text{Aut}(T)$ is a group homomorphism. If G is a torsion-free finitely generated abelian group, with generators g_1, \dots, g_n , for some positive integer n , then the skew group ring $T \rtimes_{\sigma} G$ equals the *skew Laurent polynomial ring* $T[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}; \sigma_1, \dots, \sigma_n]$, where $\sigma_i := \sigma_{g_i}$, for $i = 1, \dots, n$. In that case, T is called σ -*simple* if there is no ideal I of T , other than $\{0\}$ and T , for which $\sigma_i(I) \subseteq I$, for $i \in \{1, \dots, n\}$.

Theorem 3 (Jordan [6], Voskoglou [20]). *A skew Laurent polynomial ring*

$$T[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}; \sigma_1, \dots, \sigma_n]$$

is simple if and only if T is σ -simple and there do not exist $u \in \cap_{i=1}^n (T^{\times})^{\sigma_i}$ and a non-zero $(m_1, \dots, m_n) \in \mathbb{Z}^n$ such that for every $t \in T$, the relation $(\sigma_1^{m_1} \circ \dots \circ \sigma_n^{m_n})(t) = utu^{-1}$ holds.

The main goal of this article is to prove the following non-associative version of Theorem 1.

Theorem 4. *If a non-associative unital ring is graded by a hypercentral group, then the ring is simple if and only if it is graded simple and the center of the ring is a field.*

The secondary goal of this article is to use Theorem 4 to obtain, on the one hand, non-associative versions of Theorem 2 and Theorem 3 (see Theorem 42 and Theorem 48), and, on the other hand, a new proof of a classical result by McCrimmon concerning Cayley-Dickson doublings (see Theorem 52); recall that these algebras are non-associative and, in a natural way, graded. Here is an outline of this article.

In Section 2, we gather some well-known facts from non-associative ring and module theory that we need in the sequel. In particular, we state our conventions concerning modules over non-associative rings and what a basis should mean in that situation. In Section 3, we recall some notations and results concerning the free magma on a set, for use in subsequent sections. In Section 4, we show that Theorem 1 can be extended to non-associative rings by proving Theorem 4. In Section 5, we apply Theorem 4 to non-associative crossed products and generalize Theorem 2 and Theorem 3 (see Theorem 42 and Theorem 48). In Section 6, we show that a classical result of McCrimmon (see Theorem 52) can be deduced via Theorem 4.

2. NON-ASSOCIATIVE RINGS

In this section, we recall some notions from non-associative ring theory that we need in subsequent sections. Although the results stated in this section are presumably rather well-known, we have, for the convenience of the reader, nevertheless chosen to include proofs of these statements.

Throughout this section, R denotes a non-associative ring. By this we mean that R is an additive abelian group in which a multiplication is defined, satisfying left and right distributivity. We always assume that R is unital and that the multiplicative identity of R is denoted by 1. The term "non-associative" should be interpreted as "not necessarily

associative". Therefore all associative rings are non-associative. If a ring is not associative, we will use the term "not associative ring".

By a *left module* over R we mean an additive group M equipped with a bi-additive map $R \times M \ni (r, m) \mapsto rm \in M$. A subset B of M is said to be a *basis* for M , if for every $m \in M$, there are unique $r_b \in R$, for $b \in B$, such that $r_b = 0$ for all but finitely many $b \in B$, and $m = \sum_{b \in B} r_b b$. *Right modules* over R and their bases are defined in an analogous manner.

Recall that the *commutator* $[\cdot, \cdot] : R \times R \rightarrow R$ and the *associator* $(\cdot, \cdot, \cdot) : R \times R \times R \rightarrow R$ are defined by $[r, s] = rs - sr$ and $(r, s, t) = (rs)t - r(st)$ for all $r, s, t \in R$, respectively. The *commuter* of R , denoted by $C(R)$, is the subset of R consisting of elements $r \in R$ such that $[r, s] = 0$ for all $s \in R$. The *left, middle and right nucleus* of R , denoted by $N_l(R)$, $N_m(R)$ and $N_r(R)$, respectively, are defined by $N_l(R) = \{r \in R \mid (r, s, t) = 0, \text{ for } s, t \in R\}$, $N_m(R) = \{s \in R \mid (r, s, t) = 0, \text{ for } r, t \in R\}$, and $N_r(R) = \{t \in R \mid (r, s, t) = 0, \text{ for } r, s \in R\}$. The *nucleus* of R , denoted by $N(R)$, is defined to be equal to $N_l(R) \cap N_m(R) \cap N_r(R)$. From the so-called *associator identity* $u(r, s, t) + (u, r, s)t + (u, rs, t) = (ur, s, t) + (u, r, st)$, which holds for all $u, r, s, t \in R$, it follows that all of the subsets $N_l(R)$, $N_m(R)$, $N_r(R)$ and $N(R)$ are associative subrings of R . The *center* of R , denoted by $Z(R)$, is defined to be equal to the intersection $N(R) \cap C(R)$. It follows immediately that $Z(R)$ is an associative, unital and commutative subring of R .

Proposition 5. *The following three equalities hold:*

$$Z(R) = C(R) \cap N_l(R) \cap N_m(R); \quad (1)$$

$$Z(R) = C(R) \cap N_l(R) \cap N_r(R); \quad (2)$$

$$Z(R) = C(R) \cap N_m(R) \cap N_r(R). \quad (3)$$

Proof. We only show (1). The equalities (2) and (3) are shown in a similar way and are therefore left to the reader. It is clear that $Z(R) \subseteq C(R) \cap N_l(R) \cap N_m(R)$. Now we show the reversed inclusion. Take $r \in C(R) \cap N_l(R) \cap N_m(R)$. We need to show that $r \in N_r(R)$. Take $s, t \in R$. We wish to show that $(s, t, r) = 0$, i.e. $(st)r = s(tr)$. Using that $r \in C(R) \cap N_l(R) \cap N_m(R)$ we get $(st)r = r(st) = (rs)t = (sr)t = s(rt) = s(tr)$. \square

Proposition 6. *If $r \in Z(R)$ and $s \in R$ satisfy $rs = 1$, then $s \in Z(R)$.*

Proof. Let $r \in Z(R)$ and suppose that $rs = 1$. First we show that $s \in C(R)$. To this end, take $u \in R$. Then $su = (su)1 = (su)(rs) = (r(su))s = ((rs)u)s = (1u)s = us$ and hence $s \in C(R)$. By Proposition 5, we are done if we can show $s \in N_l(R) \cap N_m(R)$. To this end, take $v \in R$. Then $s(uv) = s((1u)v) = s(((rs)u)v) = (rs)((su)v) = 1((su)v) = (su)v$ which shows that $s \in N_l(R)$. We also see that $(us)v = (us)(1v) = (us)((rs)v) = (u(rs))(sv) = (u1)(sv) = u(sv)$ which shows that $s \in N_m(R)$. \square

Proposition 7. *If R is simple, then $Z(R)$ is a field.*

Proof. We already know that $Z(R)$ is a unital commutative ring. What is left to show is that every non-zero element of $Z(R)$ has a multiplicative inverse in $Z(R)$. To this end, take a non-zero $r \in Z(R)$. Then Rr is a non-zero ideal of R . Since R is simple, this implies

that $R = Rr$. In particular, we get that there is $s \in R$ such that $1 = sr$. By Proposition 6, we get that $s \in Z(R)$ and we are done. \square

Remark 8. Denote by R^\times the set of all elements which have two-sided multiplicative inverses. Notice that R^\times is not necessarily a group.

Proposition 9. *The equality $N(R) \cap R^\times = N(R)^\times$ holds.*

Proof. The inclusion $N(R) \cap R^\times \supseteq N(R)^\times$ is clear. Now we show the reversed inclusion. Take $r, s \in R$ satisfying $r \in N(R)$ and $rs = sr = 1$. We need to show that $s \in N(R)$. Take $u, v \in R$.

First we show that $s \in N_l(R)$.

$$s(uv) = s(1uv) = s((rs)u)v = s((r(su))v) = s(r((su)v)) = (sr)((su)v) = (su)v.$$

Now we show that $s \in N_m(R)$.

$$(us)v = (us)((rs)v) = (us)(r(sv)) = ((us)r)(sv) = (u(sr))(sv) = u(sv).$$

Finally, we show that $s \in N_r(R)$.

$$(uv)s = (u(v(sr)))s = (u((vs)r))s = ((u(vs))r)s = (u(vs))(rs) = u(vs).$$

\square

3. THE FREE MAGMA ON A SET

Throughout this section, X denotes an infinite set and R denotes a non-associative ring. Recall that we can define the *free magma on X* , denoted by $M(X)$, by induction on the length of *words* in the following way (for more details, see e.g. [19, Definition 21B.1]). We call the elements of X words of length 1. If α and β are words of length m and n respectively, then $(\alpha)(\beta)$ is a word of length $m + n$. Furthermore, two words $(\alpha)(\beta)$ and $(\alpha')(\beta')$ are considered equal precisely when $\alpha = \alpha'$ and $\beta = \beta'$. Note that in each word α there are only a finite number of letters x_1, \dots, x_n , where x_1 is the first letter (to the left) and x_n is the last letter (to the right). Then we say that α is of *length n* , and that α depends on x_1, \dots, x_n which we will indicate by writing $\alpha = \alpha(x_1, \dots, x_n)$. In that case, if $r_1, \dots, r_n \in R$, then we say that $\alpha(r_1, \dots, r_n) \in R$ is a *specialization* of α . We say that a word $\alpha(x_1, \dots, x_n)$ is *linear* if for each $i \in \{1, \dots, n\}$, the letter x_i occurs exactly once in α .

Example 10. Take distinct elements $a, b, c, d \in X$ and define the words

$$\alpha(a, b, c, d) = a((bc)d), \quad \beta(a, b, c, d) = (a(bc))d \quad \text{and} \quad \gamma(a, b, c, b) = (ab)(cb).$$

Notice that α and β are linear words, but $\alpha \neq \beta$. The word γ is not linear.

Proposition 11. *Every specialization $\alpha(r_1, \dots, r_n)$ of a word $\alpha(x_1, \dots, x_n)$ equals a specialization $\alpha'(r_1, \dots, r_m)$ of a linear word $\alpha'(x'_1, \dots, x'_n)$.*

Proof. Use induction over the length of α . \square

Proposition 12. *If A is a subset of R , then the ideal of R generated by A , denoted by $\langle A \rangle$, equals the additive group generated by specializations $\alpha(r_1, \dots, r_n)$, where $r_i \in R$, for $i \in \{1, \dots, n\}$, of linear words $\alpha(x_1, \dots, x_n)$, such that at least one of the r_i 's belongs to A .*

Proof. Let I denote the additive group generated by the specializations $\alpha(r_1, \dots, r_n)$, where $r_i \in R$, for $i \in \{1, \dots, n\}$, of linear words $\alpha(x_1, \dots, x_n)$, such that at least one of the r_i 's belongs to A . We wish to show that $I = \langle A \rangle$.

First we show that $I \supseteq \langle A \rangle$. To this end, it is enough to show that I is an ideal of R containing A . Take $r \in R$, a linear word $\alpha(x_1, \dots, x_n)$ and $r_1, \dots, r_n \in R$ such that at least one of the r_i 's belongs to A . Take $x \in X \setminus \{x_1, \dots, x_n\}$ and form the word $\alpha' = (\alpha)(x)$. Then α' is linear and $\alpha(r_1, \dots, r_n)r = \alpha'(r_1, \dots, r_n, r) \in I$. In a similar manner one can show that $r\alpha(r_1, \dots, r_n) \in I$. Therefore, I is an ideal of R . Take $a \in A$ and let $x \in X$. By specializing the linear word $\alpha(x) = x$ with $x = a$, it follows that $a \in I$. Therefore $A \subseteq I$.

Now we show that $I \subseteq \langle A \rangle$. By additivity it is enough to show this inclusion for specializations of linear words. We will show this by induction over the length of words. The base case follows immediately since specializations of words of length 1 are just elements in A . Now we show the induction step. Let n be an integer greater than 1. Suppose that the inclusion holds for linear words of length less than n , and let γ be an arbitrary linear word of length n . Then we can write $\gamma = (\alpha)(\beta)$ for some linear words α and β , both of length less than n . If, for a specialization $\gamma(r_1, \dots, r_n)$, there is some r_i which belongs to A , then that element must appear in either of the corresponding specializations of α and β . Hence, by the induction hypothesis, the specialization of α belongs to $\langle A \rangle$ or the specialization of β belongs to $\langle A \rangle$. In either case, since $\langle A \rangle$ is an ideal, the specialization $\gamma(r_1, \dots, r_n)$ belongs to $\langle A \rangle$. Hence $I \subseteq \langle A \rangle$. \square

4. SIMPLICITY

At the end of this section, we show Theorem 4. We adapt the approach taken by Jespers [5] to the non-associative situation. Throughout this section R denotes a non-associative unital ring which is graded by a group G . Furthermore, X denotes an infinite set. Recall that the elements of $h(R) := \cup_{g \in G} R_g$ are called *homogeneous*. A specialization $\alpha(r_1, \dots, r_n)$ of a word $\alpha(x_1, \dots, x_n)$ is said to be homogeneous if all the r_i 's are homogeneous.

Proposition 13. *To every specialization $\alpha(r_1, \dots, r_n)$ of a word $\alpha(x_1, \dots, x_n)$ there is a linear word $\alpha'(x'_1, \dots, x'_n)$ such that $\alpha(r_1, \dots, r_n)$ is a sum of homogeneous specializations of α' .*

Proof. By Proposition 11, there is a specialization $\alpha'(r_1, \dots, r_n)$ of a linear word $\alpha'(x'_1, \dots, x'_n)$ such that $\alpha(r_1, \dots, r_n) = \alpha'(r_1, \dots, r_n)$. Since R is graded, we can, for every $i \in \{1, \dots, n\}$, write $r_i = \sum_{g \in \text{Supp}(r_i)} (r_i)_g$. Put $H = \text{Supp}(r_1) \times \dots \times \text{Supp}(r_n)$. By linearity of α' , we get that

$$\alpha(r_1, \dots, r_n) = \alpha'(r_1, \dots, r_n) = \sum_{(g_1, \dots, g_n) \in H} \alpha'((r_1)_{g_1}, \dots, (r_n)_{g_n})$$

which is a sum of homogeneous specializations of α' . \square

Proposition 14. *If A is a subset of $h(R)$, then $\langle A \rangle$ is a graded ideal which equals the additive group generated by homogeneous specializations $\alpha(r_1, \dots, r_n)$ of linear words $\alpha(x_1, \dots, x_n)$ such that at least one of the r_i 's belongs to A .*

Proof. This follows from Proposition 12 and (the proof of) Proposition 13. \square

The following is a non-associative analogue of [12, Proposition 1.1.1].

Proposition 15. *The following holds:*

- (a) $1 \in R_e$. In particular, $R_e \neq \{0\}$.
- (b) if $r \in R^\times$ is homogeneous, then r^{-1} is homogeneous of degree $\deg(r)^{-1}$.

Proof. (a) Let $1 = \sum_{g \in G} 1_g$. For $h \in G$ and $r_h \in R_h$ we now get that $r_h = r_h \cdot 1 = \sum_{g \in G} r_h 1_g$. Therefore we get that $r_h = r_h 1_e$ and $r_h 1_g = 0$ for every $g \in G \setminus \{e\}$. In particular, for any $g \in G \setminus \{e\}$, we get that $1_g = 1 \cdot 1_g = \sum_{h \in G} 1_h 1_g = 0$. Therefore, we get that $1 = 1_e \in R_e$.

(b) Suppose that $r \in R_g$, for some $g \in G$, and that $rs = sr = 1$, for some $s \in R$. From (a) we get that $1 \in R_e$. Hence $1 = 1_e = (rs)_e = rs_{g^{-1}}$ and $1 = 1_e = (sr)_e = s_{g^{-1}}r$. This shows that $r^{-1} = s_{g^{-1}} \in R_{g^{-1}}$. \square

Proposition 16. *If R is graded simple and for some $g \in G$, the additive group R_g is non-zero, then for any non-zero $r \in h(R)$, there is a homogeneous specialization $\alpha(r_1, \dots, r_n)$ of a linear word $\alpha(x_1, \dots, x_n)$ such that $\alpha(r_1, \dots, r_n)$ is a non-zero element in R_g and at least one of the r_i 's equals r .*

Proof. Put $A = \{r\}$. By Proposition 14, we get that $\langle A \rangle$ is a non-zero graded ideal of R . Since R is graded simple, we therefore get that $\langle A \rangle = R$. In particular, this implies that $R_g = \langle A \rangle_g$. By Proposition 14 again and the fact that R_g is non-zero it follows that there is a non-zero homogeneous specialization $\alpha(r_1, \dots, r_n)$ of a linear word $\alpha(x_1, \dots, x_n)$ such that $\alpha(r_1, \dots, r_n)$ is a non-zero element in R_g and at least one of the r_i belongs to $A = \{r\}$. \square

Definition 17. Suppose that I is an ideal of R . Following Jespers [5], we say that a subset M of G satisfies the *minimal support condition* (MS), with respect to I , if there exists a non-zero $r \in I$ with $\text{Supp}(r) = M$, and there is no proper subset N of M such that $N = \text{Supp}(s)$ for some non-zero $s \in I$.

Proposition 18. *Suppose that $\alpha(r_1, \dots, r_n) \in R_h$, for some $h \in G$, is a non-zero homogeneous specialization of a linear word $\alpha(x_1, \dots, x_n)$, where for each $i \in \{1, \dots, n\}$, $r_i \in R_{h_i}$, for some $h_i \in G$. If $z_i \in Z(G)$ and $s_i \in R_{h_i z_i}$, for $i \in \{1, \dots, n\}$, then $\alpha(s_1, \dots, s_n) \in R_{h z_1 \dots z_n}$.*

Proof. We will show this by induction over the length l of α . First we show the base case $l = 1$. Then $\alpha = x_1$ with corresponding non-zero specialization $\alpha(r_1) = r_1 \in R_h \cap R_{h_1}$. But then $h = h_1$ and thus $\alpha(s_1) = s_1 \in R_{h_1 z_1} = R_{h z_1}$. Now we show the induction step. Suppose that $l > 1$ and that the claim holds for words of length less than l . Let $\alpha(x_1, \dots, x_l)$ be a linear word and suppose that $\alpha = (\beta)(\gamma)$ where $\beta = \beta(x_1, \dots, x_k)$ and

$\gamma = \gamma(x_{k+1}, \dots, x_l)$, for some suitable k . Moreover, suppose that for each $i \in \{1, \dots, l\}$, $r_i \in R_{h_i}$, for some $h_i \in G$, and that $z_i \in Z(G)$ and $s_i \in R_{h_i z_i}$.

Put $h' = \text{Supp}(\beta(r_1, \dots, r_k))$ and $h'' = \text{Supp}(\gamma(r_{k+1}, \dots, r_l))$. Notice that

$$h = \text{Supp}(\alpha(r_1, \dots, r_l)) = h'h''.$$

By the induction hypothesis, we get that

$$\alpha(s_1, \dots, s_l) = \beta(s_1, \dots, s_k)\gamma(s_{k+1}, \dots, s_l) \in R_{h'z_1 \dots z_k h''z_{k+1} \dots z_l} = R_{h'h''z_1 \dots z_k z_{k+1} \dots z_l} = R_{hz_1 \dots z_l}.$$

□

Definition 19. Let M be a subset of G and let I be a subset of R . We say that M is an I -set if there is $r \in I$ with $\text{Supp}(r) = M$. We say that M is $Z(G)$ -homogeneous if M is a subset of a coset of $Z(G)$ in G .

Remark 20. Notice that all I -sets are necessarily finite. The same conclusion holds for subsets of G satisfying MS.

Proposition 21. Suppose that R is graded simple, I is a non-zero $G/Z(G)$ -graded ideal of R , M is a non-empty $Z(G)$ -homogeneous subset of G and that $g \in G$ satisfies $R_{gM} \neq \{0\}$. The following holds:

- (a) if M is an I -set, then there is a non-empty I -set N such that $N \subseteq gM$.
- (b) if M satisfies MS, then gM satisfies MS.

Proof. Since $R_{gM} \neq \{0\}$, there is $h \in M$ with $R_{gh} \neq \{0\}$. Put $m = |M|$. Since M is $Z(G)$ -homogeneous it follows that $M = \{hz_1, \dots, hz_m\}$ for distinct $z_1, \dots, z_m \in Z(G)$ with $z_1 = e$.

(a) Suppose that M is an I -set. Then there is a non-zero $r \in I$ with $\text{Supp}(r) = M$. By Proposition 16, there is a non-zero homogeneous specialization $\alpha(r_1, \dots, r_n)$ of a linear word $\alpha(x_1, \dots, x_n)$ such that $\alpha(r_1, \dots, r_n) \in R_{gh}$ and $r_i = r_h$ for some $i \in \{1, \dots, n\}$. By Proposition 18, we get that

$$\begin{aligned} I \setminus \{0\} \ni s &:= \alpha(r_1, \dots, r_{i-1}, r, r_{i+1}, \dots, r_n) \\ &= \sum_{j=1}^m \alpha(r_1, \dots, r_{i-1}, r_{hz_j}, r_{i+1}, \dots, r_n) \in \oplus_{j=1}^m R_{ghz_j} = R_{gM} \end{aligned}$$

In particular, if we put $N = \text{Supp}(s)$, we get that N is a non-empty I -set with $N \subseteq gM$.

(b) Suppose that M satisfies MS. Using (a) we get that there is a non-empty I -set N with $N \subseteq gM$. Suppose that there is a non-empty I -set P with $P \subseteq N$. Since M is an I -set and $g^{-1}P \subseteq g^{-1}N \subseteq g^{-1}gM = M$, we conclude that $R_{g^{-1}P} \neq \{0\}$. Therefore, by (a), there is a non-empty I -set Q with $Q \subseteq g^{-1}P$. But then $Q \subseteq M$, which by MS of M implies that $Q = M$. Therefore $M \subseteq g^{-1}P \subseteq M$ and hence $P = N = gM$. This proves that gM satisfies MS with respect to I . □

Proposition 22. Suppose that R is graded simple, I is a non-zero $G/Z(G)$ -graded ideal of R and that $r \in R_{Z(G)}$ has the property that $M = \text{Supp}(r)$ satisfies MS with respect to

I. If $\alpha(s_1, \dots, s_n)$ is a non-zero homogeneous specialization of a linear word $\alpha(x_1, \dots, x_n)$, with $n \geq 2$, then, for any $g, h \in \text{Supp}(r)$, the following equality holds:

$$\begin{aligned} \alpha(s_1, \dots, s_{i-1}, r_g, s_{i+1}, \dots, s_{j-1}, r_h, s_{j+1}, \dots, s_n) &= \\ = \alpha(s_1, \dots, s_{i-1}, r_h, s_{i+1}, \dots, s_{j-1}, r_g, s_{j+1}, \dots, s_n) \end{aligned}$$

Proof. Clearly, $s_k \neq 0$, for $k \in \{1, \dots, n\}$. Put

$$t = \alpha(s_1, \dots, s_{i-1}, r, s_{i+1}, \dots, s_{j-1}, r_h, s_{j+1}, \dots, s_n)$$

and

$$t' = \alpha(s_1, \dots, s_{i-1}, r_h, s_{i+1}, \dots, s_{j-1}, r, s_{j+1}, \dots, s_n).$$

Seeking a contradiction, suppose that $t \neq t'$. Then $t - t' \in I \setminus \{0\}$ and $\text{Supp}(t - t') \subsetneq h \prod_{1 \leq k \leq n, k \neq i, j} \deg(s_k)M$. But from Proposition 21(b), we get that $h \prod_{1 \leq k \leq n, k \neq i, j} \deg(s_k)M$ satisfies MS which is a contradiction. Thus, $t = t'$. Comparing terms of degree

$$gh \prod_{1 \leq k \leq n, k \neq i, j} \deg(s_k)$$

in the last equality yields the desired conclusion. \square

Proposition 23. *If R is graded simple and I is a $G/Z(G)$ -graded ideal of R , then $I = R(I \cap R_{Z(G)} \cap Z(R))$.*

Proof. The inclusion $I \supseteq R(I \cap R_{Z(G)} \cap Z(R))$ is clear. Now we show the inclusion $I \subseteq R(I \cap R_{Z(G)} \cap Z(R))$. Take a non-zero $a \in I$.

Case 1: $\text{Supp}(a)$ satisfies MS and $a \in R_{Z(G)}$. Fix $z \in \text{Supp}(a)$. For each $g \in \text{Supp}(a)$, define a map $f_g : R \rightarrow R$ in the following way. Take $r \in R$ and linear words $\alpha_j(x_1, \dots, x_{n(j)})$, for $j \in \{1, \dots, m\}$, such that

$$r = \sum_{j=1}^m \alpha_j(r_{(1,j)}, \dots, r_{(j',j)}, \dots, r_{(n(j),j)})$$

for some $r_{(p,q)} \in h(R)$, where $r_{(j',j)} = a_z$. Put

$$f_g(r) = \sum_{j=1}^m \alpha_j(r_{(1,j)}, \dots, r_{(j',j)}, \dots, r_{(n(j),j)})$$

where $r_{(j',j)} = a_g$. Now we show that f_g is well-defined. Suppose that

$$0 = \sum_{j=1}^m \alpha_j(r_{(1,j)}, \dots, r_{(j',j)}, \dots, r_{(n(j),j)})$$

for some $r_{(p,q)} \in h(R)$, where $r_{(j',j)} = a_z$. Suppose that $\beta(x'_1, \dots, x'_p)$ is a linear word. For any $s_1, \dots, s_p \in h(R)$, with $s_q = a_g$, for some $q \in \{1, \dots, p\}$, we get, by Proposition 22, that

$$0 = \sum_{j=1}^m \beta(s_1, \dots, s_q, \dots, s_p) \alpha_j(r_{(1,j)}, \dots, r_{(j',j)}, \dots, r_{(n(j),j)})$$

$$\begin{aligned}
&= \sum_{j=1}^m \beta(s_1, \dots, a_z, \dots, s_p) \alpha_j(r_{(1,j)}, \dots, a_g, \dots, r_{(n(j),j)}) \\
&= \beta(s_1, \dots, a_z, \dots, s_p) \sum_{j=1}^m \alpha_j(r_{(1,j)}, \dots, a_g, \dots, r_{(n(j),j)}).
\end{aligned}$$

By graded simplicity of R and Proposition 14, we get that $f_g(r) = 0$. It is clear that f_g is both a left R -module map and a right R -module map. In particular, we get that $a_g = f(a_z) = a_z f_g(1)$ where $f_g(1) \in I \cap R_{Z(G)} \cap Z(R)$. Hence $a = \sum_{g \in \text{Supp}(a)} a_g = \sum_{g \in \text{Supp}(a)} a_z f_g(1) = a_z c(z, a)$ where $c(z, a) = \sum_{g \in \text{Supp}(a)} f_g(1) \in I \cap R_{Z(G)} \cap Z(R)$. Note that $c(z, a)_e = c(z, z) = 1$.

Case 2: $\text{Supp}(a)$ satisfies MS. Since I is $G/Z(G)$ -graded, we can assume that a is $Z(G)$ -homogeneous. By Proposition 15(a) and Proposition 21, we thus get that there is a non-zero $b \in I \cap R_{Z(G)}$ and $h \in \text{Supp}(a)$ such that $\text{Supp}(b) = h^{-1} \text{Supp}(a)$ satisfies MS. By case 1, (with $z = e$) we get that there is $c \in I \cap R_{Z(G)} \cap Z(R)$ such that $b = b_e c$, $c_e = 1$ and $\text{Supp}(c) = \text{Supp}(b)$. Then $a - a_h c \in I$ and $\text{Supp}(a - a_h c) \subsetneq \text{Supp}(a)$. Hence, using that $\text{Supp}(a)$ satisfies MS, we get that $a = a_h c \in R(I \cap R_{Z(G)} \cap Z(R))$.

Case 3: $\text{Supp}(a)$ does not satisfy MS. We prove the relation $a \in R(I \cap R_{Z(G)} \cap Z(R))$ by induction. Suppose that for every non-zero $b \in I$ with $\text{Supp}(b) \subsetneq \text{Supp}(a)$, the relation $b \in R(I \cap R_{Z(G)} \cap Z(R))$ holds. Take a proper non-empty subset X of $\text{Supp}(a)$ and $b \in I$ with $\text{Supp}(b) = X$ satisfying MS. Take $h \in \text{Supp}(b)$. By Case 2, there is $c \in I \cap R_{Z(G)} \cap Z(R)$ with $c_e = 1$ and $\text{Supp}(c) = h^{-1} \text{Supp}(b)$. Then $\text{Supp}(a - a_h c) \subsetneq \text{Supp}(a)$ and $a - a_h c \in I$. Hence, by the induction hypothesis, we get that $a - a_h c \in R(I \cap R_{Z(G)} \cap Z(R))$. But since obviously $a_h c \in R(I \cap R_{Z(G)} \cap Z(R))$, we finally get that $a \in R(I \cap R_{Z(G)} \cap Z(R))$. \square

Corollary 24. *If R is graded simple and $Z(R)$ is a field, then R is $G/Z(G)$ -graded simple.*

Proof. Suppose that I is a non-zero $G/Z(G)$ -graded ideal of R . By Proposition 23, I contains a non-zero element from $Z(R)$. Since $Z(R)$ is a field this implies that $I = R$. Thus R is $G/Z(G)$ -graded simple. \square

Remark 25. Recall that if G is a group with identity element e , then the ascending central series of G is the sequence of subgroups $Z_i(G)$, for non-negative integers i , defined recursively by $Z_0(G) = \{e\}$ and, given $Z_i(G)$, for some non-negative integer i , $Z_{i+1}(G)$ is defined to be the set of $g \in G$ such that for every $h \in G$, the commutator $[g, h] = ghg^{-1}h^{-1}$ belongs to $Z_i(G)$. For infinite groups this process can be continued to infinite ordinal numbers by transfinite recursion. For a limit ordinal O , we define $Z_O(G) = \cup_{i < O} Z_i(G)$. If G is hypercentral, then $Z_O(G) = G$ for some limit ordinal O . For details concerning this construction, see [18].

Proposition 26. *If G is a hypercentral group and R is a G -graded ring with the property that for each $i < O$ the ring R is $G/Z_i(G)$ -graded simple, then R is simple.*

Proof. Take a non-zero ideal J of R and a non-zero $a \in J$. We show that $\langle a \rangle = R$. Since $\cup_i Z_i(G) = G$ and $\text{Supp}(a)$ is finite, we can conclude that there is some i such that

$\text{Supp}(a) \subseteq Z_i(G)$. Then $\langle a \rangle$ is a non-zero $G/Z_i(G)$ -graded ideal of R . Since R is $G/Z_i(G)$ -graded simple, we get that $\langle a \rangle = R$, which shows that $J = R$. \square

Proof of Theorem 4. Suppose that R is graded simple and that $Z(R)$ is a field. Let Z_i , for $i \geq 0$, be the ascending central series of G (see Remark 25). By Proposition 26, we are done if we can show that R is G/Z_i -graded simple for each $i \geq 0$. The base case $i = 0$ holds since R is G -graded simple. Now we show the induction step. Suppose that the statement is true for some i , i.e. that R is G/Z_i -graded simple. By Corollary 24, we get that R is $\frac{G/Z_i}{Z(G/Z_i)}$ -graded simple. Since the center of G/Z_i equals Z_{i+1}/Z_i we get that R is $\frac{G/Z_i}{Z_{i+1}/Z_i}$ -graded simple, i.e. R is G/Z_{i+1} -graded simple, and the induction step is complete. \square

Definition 27. The gradation on R is called *faithful* (see e.g. [2]) if for any $g, h \in \text{Supp}(R)$ and any non-zero $r \in R_g$ we have that $rR_h \neq \{0\} \neq R_hr$. Recall that the *finite conjugate subgroup* of G , denoted by $\Delta(G)$, is the set of elements of G which only have finitely many conjugates (see e.g. [17]).

Proposition 28. *Suppose that R is faithfully G -graded and that $\text{Supp}(R) = G$. Then $C(R) \subseteq R_{\Delta(G)}$, and, in particular, $Z(R) \subseteq R_{\Delta(G)}$.*

Proof. Take a non-zero $r \in C(R)$ and $g \in \text{Supp}(r)$. Take $h \in G$. From the assumptions it follows that there is $s \in R_h$ with $sr_g \neq 0$. Since $sr = rs$ we get that $hg \in \text{Supp}(sr) = \text{Supp}(rs) \subseteq \text{Supp}(r)h$. Thus $hgh^{-1} \in \text{Supp}(r)$. Since $\text{Supp}(r)$ is finite, we get that $g \in \Delta(G)$. The last statement follows immediately since $Z(R) \subseteq C(R)$. \square

Corollary 29. *If R is a faithfully G -graded ring with $\text{Supp}(R) = G$, where G is a torsion-free hypercentral group, then R is simple if and only if R is graded simple and $Z(R) \subseteq R_e$.*

Proof. First we show the "if" statement. Suppose that R is graded simple and that $Z(R) \subseteq R_e$. We claim that $Z(R)$ is a field. If we assume that the claim holds, then, from Theorem 4, we get that R is simple. Now we show the claim. From Section 2, we know that $Z(R)$ is a commutative unital ring. Take a non-zero $r \in Z(R)$. Since $r \in R_e$, we get that $\langle r \rangle = Rr = rR$ is a non-zero graded ideal. By graded simplicity of R , we get that there is $s \in R$ with $rs = sr = 1$. Proposition 6 yields $s \in Z(R)$. This shows that $Z(R)$ is a field.

Now we show the "only if" statement. Suppose that R is simple. From Theorem 4, we get that R is graded simple and that $Z(R)$ is a field. We wish to show that $Z(R) \subseteq R_e$. From [10, Theorem 1] it follows that $\Delta(G) = Z(G)$. By Proposition 28, we get that $Z(R) \subseteq R_{Z(G)}$. Thus, $Z(R)$ is a $Z(G)$ -graded ring. In fact, suppose that $r \in Z(R)$ and $s \in R$. Take $g \in G$ and $h \in Z(G)$. From the equality $rs_g = s_g r$ and the fact that $\text{Supp}(r) \subseteq Z(G)$, we get that $r_h s_g = s_g r_h$. Summing over $g \in G$, we get that $r_h s = s r_h$. Thus, $r_h \in Z(R)$. It is well-known that since the group $Z(G)$ is torsion free and abelian, it can be ordered with order $<$ (see e.g. [18]). Seeking a contradiction, suppose that there is a non-zero $r \in Z(R) \cap R_g$ for some $g \in Z(G) \setminus \{e\}$. Since $Z(R)$ is a field, the element $1 + r$ must have a multiplicative inverse $s \in Z(R)$. Suppose that $s = \sum_{i=1}^n s_{g_i}$, for some $g_1, \dots, g_n \in Z(G)$ with $g_1 < \dots < g_n$ and every s_{g_i} non-zero and n chosen as small as

possible. Then $1 = (1+r)s = \sum_{i=1}^n (s_{g_i} + rs_{g_i})$. Case 1: $g > 0$. Then the element s_{g_1} , in the last sum, has the unique smallest degree. Thus, $1 = s_{g_1}$ and hence $0 = r + \sum_{i=2}^n (s_{g_i} + rs_{g_i})$. In this sum the element rs_{g_n} has the unique largest degree. Thus $rs_{g_n} = 0$. Since $r \neq 0$ and $Z(R)$ is a field, we get that $s_{g_n} = 0$. This contradicts the minimality of n . Thus $Z(R) \subseteq R_e$. Case 2: $g < 0$. This case is treated analogously to Case 1. \square

5. APPLICATION: NON-ASSOCIATIVE CROSSED PRODUCTS

In this section, we begin by recalling the (folkloristic) definitions of non-associative crossed systems and non-associative crossed products (see Definition 30 and Definition 31). Then we show that the family of non-associative crossed products appear in the class of non-associative strongly graded rings in a fashion similar to how the associative crossed products present themselves in the family of associative strongly graded rings (see Proposition 33 and Proposition 34). Thereafter, we determine the center of non-associative crossed products (see Proposition 39). At the end of this section, we use Theorem 4 to obtain non-associative versions of Theorem 2 and Theorem 3 (see Theorem 42 and Theorem 48).

Definition 30. A *non-associative crossed system* is a quadruple (T, G, σ, α) consisting of a non-associative unital ring T , a group G and maps $\sigma : G \rightarrow \text{Aut}(T)$ and $\alpha : G \times G \rightarrow N(T)^\times$ satisfying the following three conditions for any triple $g, h, s \in G$ and any $a \in T$:

- (N1) $\sigma_g(\sigma_h(a)) = \alpha(g, h)\sigma_{gh}(a)\alpha(g, h)^{-1}$;
- (N2) $\alpha(g, h)\alpha(gh, s) = \sigma_g(\alpha(h, s))\alpha(g, hs)$;
- (N3) $\sigma_e = \text{id}_T$ and $\alpha(g, e) = \alpha(e, g) = 1_T$.

Definition 31. Suppose that (T, G, σ, α) is a non-associative crossed system. The corresponding *non-associative crossed product*, denoted by $T \rtimes_\sigma^\alpha G$, is defined as the set of finite sums $\sum_{g \in G} t_g u_g$ equipped with coordinate-wise addition and multiplication defined by the bi-additive extension of the relations $(au_g)(bu_h) = a\sigma_g(b)\alpha(g, h)u_{gh}$, for $a, b \in T$ and $g, h \in G$.

Remark 32. The so-called *canonical G -gradation on $R = T \rtimes_\sigma^\alpha G$* is obtained by putting $R_g = Tu_g$, for $g \in G$.

All non-associative crossed products share the following properties.

Proposition 33. Let $R = T \rtimes_\sigma^\alpha G$ be a non-associative crossed product, equipped with its canonical G -gradation. The following two assertions hold:

- (a) for each $g \in G$, the intersection $R_g \cap N(R)^\times$ is non-empty;
- (b) R is strongly G -graded, and, in particular, faithfully graded.

Proof. (a) Take $g \in G$. From Proposition 9 it follows that it is enough to show that $u_g \in R_g \cap R^\times \cap N(R)$. Put $a = \alpha(g, g^{-1})^{-1}$ and $v = \sigma_{g^{-1}}(a)u_{g^{-1}}$. Then

$$u_g v = \sigma_g(\sigma_{g^{-1}}(a))\alpha(g, g^{-1})u_e = aa^{-1}u_e = u_e$$

and from (N2) we get $\sigma_{g^{-1}}(\alpha(g, g^{-1})) = \alpha(g^{-1}, g)$ which yields

$$vu_g = \sigma_{g^{-1}}(a)\alpha(g^{-1}, g)u_e = u_e.$$

Therefore, $u_g \in R_g \cap R^\times$. Now we show that $u_g \in N(R)$. To this end, take $a, b \in T$ and $s, t \in G$.

First we show that $u_g \in N_l(R)$. On one hand we get

$$(u_g \cdot au_s) \cdot bu_t = \sigma_g(a)\alpha(g, s)u_{gs} \cdot bu_t = \sigma_g(a)\alpha(g, s)\sigma_{gs}(b)\alpha(gs, t)u_{gst}$$

and on the other hand we get

$$\begin{aligned} u_g \cdot (au_s \cdot bu_t) &= u_g \cdot a\sigma_s(b)\alpha(s, t)u_{st} = \sigma_g(a\sigma_s(b)\alpha(s, t))\alpha(g, st)u_{gst} \\ &= \sigma_g(a)\sigma_g(\sigma_s(b))\sigma_g(\alpha(s, t))\alpha(g, st)u_{gst}. \end{aligned}$$

By (N1) and (N2), the last expression equals

$$\sigma_g(a)\alpha(g, s)\sigma_{gs}(b)\alpha(g, s)^{-1}\alpha(g, s)\alpha(gs, t)u_{gst} = \sigma_g(a)\alpha(g, s)\sigma_{gs}(b)\alpha(gs, t)u_{gst}.$$

Now we show that $u_g \in N_m(R)$. On one hand we get

$$(au_s \cdot u_g) \cdot bu_t = a\alpha(s, g)u_{sg} \cdot bu_t = a\alpha(s, g)\sigma_{sg}(b)\alpha(sg, t)u_{sgt}$$

and on the other hand we get

$$\begin{aligned} au_s \cdot (u_g \cdot bu_t) &= au_s \cdot \sigma_g(b)\alpha(g, t)u_{gt} = a\sigma_s(\sigma_g(b)\alpha(g, t))\alpha(s, gt)u_{sgt} \\ &= a\sigma_s(\sigma_g(b))\sigma_s(\alpha(g, t))\alpha(s, gt)u_{sgt}. \end{aligned}$$

By (N1) and (N2), the last expression equals

$$a\alpha(s, g)\sigma_{sg}(b)\alpha(s, g)^{-1}\alpha(s, g)\alpha(sg, t)u_{sgt} = a\alpha(s, g)\sigma_{sg}(b)\alpha(sg, t)u_{sgt}.$$

Finally, we show that $u_g \in N_r(R)$. On one hand we get

$$(au_s \cdot bu_t) \cdot u_g = a\sigma_s(b)\alpha(s, t)u_{st} \cdot u_g = a\sigma_s(b)\alpha(s, t)\alpha(st, g)u_{stg}$$

and on the other hand, using (N2), we get

$$\begin{aligned} au_s \cdot (bu_t \cdot u_g) &= au_s \cdot b\alpha(t, g)u_{tg} = a\sigma_s(b\alpha(t, g))\alpha(s, tg)u_{stg} \\ &= a\sigma_s(b)\sigma_s(\alpha(t, g))\alpha(s, tg)u_{stg} = a\sigma_s(b)\alpha(s, t)\alpha(st, g)u_{stg}. \end{aligned}$$

(b) Take $g, h \in G$ and $t \in T$. Clearly, the inclusion $R_g R_h \subseteq R_{gh}$ holds. Notice that $tu_{gh} = t\alpha(g, h)^{-1}\alpha(g, h)u_{gh} = (t\alpha(g, h)^{-1}u_g)u_h$. This shows that $R_{gh} \subseteq R_g R_h$. \square

In fact, the property of Proposition 33(a) characterizes non-associative crossed products.

Proposition 34. *Every G -graded non-associative unital ring R with the property that for each $g \in G$, the intersection $R_g \cap N(R)^\times$ is non-empty, is a non-associative crossed product of the form $T \rtimes_\sigma^\alpha G$, associated with a non-associative crossed system (T, G, σ, α) .*

Proof. For each $g \in G$, take $u_g \in R_g \cap N(R)^\times$. Put $T = R_e$ and, supported by Proposition 15(a), choose $u_e = 1$.

First we show that R , considered as a left (or right) T -module, is free with $\{u_g\}_{g \in G}$ as a basis. From the gradation it follows that it is enough to show that for each $g \in G$, R_g , considered as a left (or right) T -module is free with u_g as a basis. Since $u_g \in R^\times \cap N(R)$ it follows that the left T -module Tu_g (or the right T -module $u_g T$) is free. Take $g \in G$. From

Proposition 15(b) it follows that $u_g^{-1} \in R_{g^{-1}}$. Thus, $R_g = R_g u_g^{-1} u_g \subseteq T u_g \subseteq R_g$. Hence, $R_g = T u_g$. Analogously, $R_g = u_g u_g^{-1} R_g \subseteq u_g T \subseteq R_g$ which implies that $R_g = u_g T$.

Define $\sigma : G \rightarrow \text{Aut}(T)$ by the relation $\sigma_g(a) = u_g a u_g^{-1}$, for $g \in G$ and $a \in T$. Define $\alpha : G \times G \rightarrow N(T)^\times$ by the relations $\alpha(g, h) = u_g u_h u_{gh}^{-1}$, for $g, h \in G$. Now we check conditions (N1)–(N3) of Definition 30. To this end, take $g, h, s \in G$ and $a, b \in T$.

First we show (N1):

$$\begin{aligned} \sigma_g(\sigma_h(a)) &= u_g u_h a u_h^{-1} u_g^{-1} = u_g u_h u_{gh}^{-1} u_{gh} a u_{gh}^{-1} u_{gh} u_h^{-1} u_g^{-1} \\ &= \alpha(g, h) \sigma_{gh}(a) \alpha(g, h)^{-1} \end{aligned}$$

Next we show (N2):

$$\begin{aligned} \alpha(g, h) \alpha(gh, s) &= u_g u_h u_{gh}^{-1} u_{gh} u_s u_{ghs}^{-1} = u_g u_h u_s u_{ghs}^{-1} = u_g u_h u_s u_{hs}^{-1} u_{hs} u_{ghs}^{-1} \\ &= u_g \alpha(h, s) u_g^{-1} u_g u_{hs} u_{ghs}^{-1} = \sigma_g(\alpha(h, s)) \alpha(g, hs) \end{aligned}$$

Finally we show (N3): $\sigma_e = \text{id}_T$ is immediate. Moreover, we get

$$\alpha(g, e) = u_g u_e u_g^{-1} = 1 = u_e u_g u_g^{-1} = \alpha(e, g).$$

We conclude our proof by showing that the multiplication in R is compatible with the crossed product multiplication rule:

$$(a u_g)(b u_h) = a u_g b u_g^{-1} u_g u_h = a \sigma_g(b) u_g u_h u_{gh}^{-1} u_{gh} = a \sigma_g(b) \alpha(g, h) u_{gh}.$$

□

Remark 35. Proposition 33 and Proposition 34 are non-associative generalizations of [12, Proposition 1.4.1] and [12, Proposition 1.4.2], respectively.

Proposition 36. *A non-associative crossed product $T \rtimes_\sigma^\alpha G$ is associative if and only if T is associative.*

Proof. Put $R = T \rtimes_\sigma^\alpha G$. Since T is a subring of R , the "only if" statement is clear. Now we show the "if" statement. Suppose that T is associative. From Section 2 we know that $N(R)$ is a subring of R . Thus, from the proof of Proposition 33(a), it follows that it is enough to show that $T \subseteq N(R)$. To this end, take $a, b, t \in T$ and $g, h \in G$.

First we show that $t \in N_l(R)$. We get

$$t \cdot (a u_g \cdot b u_h) = t(a \sigma_g(b) \alpha(g, h) u_{gh}) = t(a \sigma_g(b) \alpha(g, h)) u_{gh}$$

and since T is associative, the last expression equals

$$(ta)(\sigma_g(b) \alpha(g, h)) u_{gh} = (ta u_g) \cdot (b u_h) = (t \cdot a u_g) \cdot (b u_h).$$

Next we show that $t \in N_m(R)$. We get

$$a u_g \cdot (t \cdot b u_h) = a u_g \cdot (t b u_h) = a \sigma_g(t b) \alpha(g, h) u_{gh} = a(\sigma_g(t) \sigma_g(b)) \alpha(g, h) u_{gh}$$

and since T is associative, the last expression equals

$$(a \sigma_g(t)) \sigma_g(b) \alpha(g, h) u_{gh} = (a \sigma_g(t) u_g) \cdot (b u_h) = (a u_g \cdot t) \cdot (b u_h).$$

Finally, we show that $t \in N_r(R)$. We get

$$(au_g \cdot bu_h) \cdot t = (a\sigma_g(b)\alpha(g, h)u_{gh}) \cdot t = (a\sigma_g(b)\alpha(g, h)) \cdot \sigma_{gh}(t)u_{gh}$$

and using (N1) and the fact that T is associative, the last expression equals

$$\begin{aligned} a(\sigma_g(b)\alpha(g, h)\sigma_{gh}(t))u_{gh} &= a(\sigma_g(b)\alpha(g, h)(\alpha(g, h)^{-1}\sigma_g(\sigma_h(t))\alpha(g, h)))u_{gh} \\ &= a(\sigma_g(b)\sigma_g(\sigma_h(t)))\alpha(g, h)u_{gh} = a\sigma_g(b\sigma_h(t))\alpha(g, h)u_{gh} \\ &= (au_g) \cdot (b\sigma_h(t)u_h) = (au_g) \cdot (bu_h \cdot t). \end{aligned}$$

□

Definition 37. Let (T, G, σ, α) be a non-associative crossed system. An ideal I of T is said to be G -invariant if for every $g \in G$, the inclusion $\sigma_g(I) \subseteq I$ holds. The ring T is said to be G -simple if the only G -invariant ideals of T are $\{0\}$ and T itself.

Proposition 38. A non-associative crossed product $T \rtimes_\sigma^\alpha G$ is graded simple, with respect to its canonical G -gradation, if and only if T is G -simple.

Proof. Put $R = T \rtimes_\sigma^\alpha G$. First we show the "only if" statement. Suppose that R is graded simple. Let J be a non-zero G -invariant ideal of T . Put $I = JR$. Then I is a non-zero graded ideal of R . From graded simplicity of R it follows that $I = R$. Therefore, $J = T$. Now we show the "if" statement. Suppose that T is G -simple. Let I be a non-zero graded ideal of R . Consider the ideal $J = I \cap T$ of T . From the proof of Proposition 33(a) we notice that $u_g \in N(R)^\times$, for each $g \in G$, and hence J is non-zero. Take a non-zero $t \in J$. From the equalities $\sigma_g(t)\alpha(g, g^{-1})^{-1} = u_g t u_{g^{-1}}$, for $g \in G$, it follows that J is G -invariant. By G -simplicity of T we get that $J = I \cap T = T$. In particular, $1 \in I$ and hence $I = R$. □

Proposition 39. The center of a non-associative crossed product $T \rtimes_\sigma^\alpha G$ equals the set of elements of the form $\sum_{g \in G} t_g u_g$, with $t_g \in T$, such that for all $g, h \in G$ and all $t \in T$, the following four properties hold:

- (i) $tt_g = t_g \sigma_g(t)$;
- (ii) $t_{hgh^{-1}} = \sigma_h(t_g)\alpha(h, g)\alpha(hgh^{-1}, h)^{-1}$;
- (iii) $t_g \in N(T)$;
- (iv) if $g \notin \Delta(G)$, then $t_g = 0$.

Proof. Put $R = T \rtimes_\sigma^\alpha G$ and let $r = \sum_{g \in G} t_g u_g \in R$. Take $s, t \in T$ and $h \in G$.

Suppose first that $r \in Z(R)$. Property (i) follows from the equality $tr = rt$. Property (ii) follows from the equality $ru_h = u_h r$. Property (iii) follows from the equalities $(st)r = s(tr)$, $(sr)t = s(rt)$ and $(rs)t = r(st)$. Property (iv) follows from property (ii) above.

Now suppose that r satisfies properties (i), (ii), (iii) and (iv). We wish to show that $r \in Z(R)$. From (i) and (ii), and the fact that $u_h \in N(R)$ (see the proof of Proposition 33(a)), we get that $r \in C(R)$. From (iii), and the fact that $u_g \in N(R)$, for all $g \in G$, we get that $r \in N(R)$. □

Proposition 40. If $T \rtimes_\sigma^\alpha G$ is a non-associative crossed product, where T is G -simple, then $Z(T)^G := \{t \in Z(T) \mid \sigma_g(t) = t, \forall g \in G\}$ is a field.

Proof. Clearly, $Z(T)^G$ is a commutative unital ring. Now we show that every non-zero element in $Z(T)^G$ has a multiplicative inverse. To this end, put $R = T \rtimes_\sigma^\alpha G$. Take a non-zero $t \in Z(T)^G$. Put $I = tT = Tt$. Then I is a non-zero G -invariant ideal of T . By G -simplicity of T , we get that $I = Z(T)^G$. In particular, there is $s \in T$ such that $st = ts = 1$. By Proposition 6, we get that $s \in Z(T)$. Take $g \in G$. Then $\sigma_g(s) = \sigma_g(s)1 = \sigma_g(s)ts = \sigma_g(s)\sigma_g(t)s = \sigma_g(st)s = \sigma_g(1)s = 1s = s$. Hence, $s \in Z(T)^G$. \square

Lemma 41. *If G is a torsion-free hypercentral group, then $\Delta(G) = Z(G)$.*

Proof. See [10, Theorem 1]. \square

Theorem 42. *If $T \rtimes_\sigma^\alpha G$ is a non-associative crossed product, where G is a torsion-free hypercentral group, then the following three conditions are equivalent:*

- (i) $T \rtimes_\sigma^\alpha G$ is simple;
- (ii) T is G -simple and $Z(T \rtimes_\sigma^\alpha G) = Z(T)^G$;
- (iii) T is G -simple and there do not exist $u \in T^\times$ and $g \in Z(G) \setminus \{e\}$ such that for every $h \in G$ and every $t \in T$, the relations $\sigma_h(u) = u\alpha(g, h)\alpha(h, g)^{-1}$ and $\sigma_g(t) = u^{-1}tu$ hold.

Proof. Put $R = T \rtimes_\sigma^\alpha G$ with its canonical G -gradation.

(i) \Rightarrow (ii): It follows from Proposition 38 that T is G -simple. From Corollary 29, we get that $Z(R) \subseteq T$. From Proposition 39, we thus get that $Z(R) = Z(T)^G$.

(ii) \Rightarrow (i): We notice that $Z(R) \subseteq R_e = T$. Hence, the desired conclusion follows from Proposition 38 and Corollary 29.

(iii) \Rightarrow (i): We claim that $Z(R) \subseteq R_e$. If we assume that the claim holds, then (i) follows from Proposition 38 and Corollary 29. Now we show the claim. Take $r = \sum_{h \in G} t_h u_h \in Z(R)$. Seeking a contradiction, suppose that there is some $g \in \text{Supp}(r) \setminus \{e\}$. By Proposition 39 and Lemma 41, it follows that $g \in Z(G)$. Put $u = t_g$. From Proposition 39(i), we get that $Tu = uT$ is a non-zero G -invariant ideal of T . Thus, from G -simplicity of T , we get that $u \in T^\times$. This contradicts Proposition 39(i)–(ii).

(i) \Rightarrow (iii): Seeking a contradiction, suppose that (iii) fails to hold. Then there is some $u \in T^\times$ and $g \in Z(G) \setminus \{e\}$ such that for every $h \in G$ and every $t \in T$, the relations $\sigma_h(u) = u\alpha(g, h)\alpha(h, g)^{-1}$ and $\sigma_g(t) = u^{-1}tu$ hold. By Proposition 39, the element $1 + uu_g$ belongs to $Z(R) \setminus R_e$. This contradicts Corollary 29. \square

Lemma 43. *Suppose that $T \rtimes_\sigma^\alpha G$ is a non-associative crossed product and that H is a subgroup of G . Let σ' and α' be the restriction of σ and α to H and $H \times H$, respectively. Then $T \rtimes_{\sigma'}^{\alpha'} H$ is a non-associative crossed product and the map $\pi : T \rtimes_\sigma^\alpha G \rightarrow T \rtimes_{\sigma'}^{\alpha'} H$, defined by $\pi(\sum_{g \in G} t_g u_g) = \sum_{h \in H} t_h u_h$, is a $T \rtimes_{\sigma'}^{\alpha'} H$ -bimodule homomorphism. Hence, if $r \in (T \rtimes_{\sigma'}^{\alpha'} H) \cap (T \rtimes_\sigma^\alpha G)^\times$, then $r \in (T \rtimes_{\sigma'}^{\alpha'} H)^\times$.*

Proof. The argument used for the proofs of Lemma 1.2 and Lemma 1.4 in [17], for associative group rings, can easily be adapted to the case of non-associative crossed products. \square

Definition 44. If a non-associative crossed product $T \rtimes_\sigma^\alpha G$ satisfies $\alpha(g, h) = 1$, for all $g, h \in G$, then we call it a *non-associative skew group ring* and denote it by $T \rtimes_\sigma G$; in this

case $\sigma : G \rightarrow \text{Aut}(T)$ is a group homomorphism. On the other hand, if a non-associative crossed product $T \rtimes_\sigma^\alpha G$ satisfies $\sigma_g = \text{id}_T$, for all $g \in G$, then we call it a *non-associative twisted group ring* and denote it by $T \rtimes^\alpha G$.

In the sequel, we shall refer to the following set

$$L = \{g \in G \mid \sigma_g \text{ acts as conjugation by an invertible element of } N(T)^G\}.$$

Lemma 45. *Suppose that $T \rtimes_\sigma G$ is a non-associative skew group ring, where G is an abelian group and T is G -simple. Then $Z(T \rtimes_\sigma G)$ equals a twisted group ring $F \rtimes^\alpha L$, where $F = T^G \cap Z(T)$.*

Proof. Put $R = T \rtimes_\sigma G$. Since G is abelian, it follows from Proposition 39 that $Z(R) = \bigoplus_{g \in G} T_g u_g$, where for each $g \in G$, T_g is the set of $t_g \in T^G \cap N(T)$ satisfying $t_g \sigma_g(t) = t t_g$, for $t \in T$. From G -simplicity of T it follows that each non-zero $t_g \in T_g$ is invertible. Thus, $Z(R) = \bigoplus_{g \in L} T_g u_g$. Note that $F = T_e$. Moreover, if $g \in L$ and $t_g, s_g \in T_g$ are non-zero, then, for each $t \in T$, we have that $\sigma_g^{-1} \sigma_g(t) = t$. Thus, $s_g t_g^{-1} t t_g s_g^{-1} = t$ and hence $t t_g s_g^{-1} = t_g s_g^{-1} t$ from which it follows that $t_g s_g^{-1} \in F$. Therefore $t_g \in F s_g$. We can thus write $Z(R) = \bigoplus_{g \in L} F d_g$, where $d_g = s_g u_g$. Take $g, h \in L$. Then it is clear that $d_g d_h = \alpha(g, h) d_{gh}$, where $\alpha(g, h) = s_g s_h s_{gh}^{-1}$. Now we show that $\alpha(g, h) \in F$. Since $s_g, s_h, s_{gh} \in T^G$, it follows that $\alpha(g, h) \in T^G$. Next we show that $\alpha(g, h) \in C(T)$. Take $t \in T$. Then $t \alpha(g, h) = t s_g s_h s_{gh}^{-1} = s_g \sigma_g(t) s_h s_{gh}^{-1} = s_g s_h \sigma_{hg}(t) s_{gh}^{-1} = s_g s_h \sigma_{gh}(t) s_{gh}^{-1} = s_g s_h s_{gh}^{-1} t = \alpha(g, h) t$. Since we already know that $s_g, s_h, s_{gh}^{-1} \in N(T)$, we thus get that $\alpha(g, h) \in Z(T)$. Finally, we show (N2) of Definition 30. To this end, take $g, h, p \in G$. Then

$$\begin{aligned} \alpha(g, h) \alpha(gh, p) &= s_g s_h s_{gh}^{-1} s_{gh} s_p s_{ghp}^{-1} = s_g s_h s_p s_{ghp}^{-1} = s_g \sigma_g(s_h s_p) s_{ghp}^{-1} = s_h s_p s_g s_{ghp}^{-1} \\ &= s_h s_p \sigma_{hp}(s_g) s_{ghp}^{-1} = s_h s_p s_{hp}^{-1} s_g s_{hgp} s_{ghp}^{-1} = \alpha(h, p) \alpha(g, hp). \end{aligned}$$

Thus $Z(R) = F \rtimes^\alpha L$. □

The next result is a non-associative generalization of [1, Proposition 5.7].

Proposition 46. *Suppose that $T \rtimes_\sigma G$ is a non-associative skew group ring, where G is an abelian group.*

- (a) *$T \rtimes_\sigma G$ is simple if and only if T is G -simple, $Z(T \rtimes_\sigma^\alpha G)$ is a domain, and L is a torsion group.*
- (b) *If T is commutative, then $T \rtimes_\sigma G$ is simple if and only if T is G -simple and $L = \{e\}$.*

Proof. Put $R = T \rtimes_\sigma G$.

(a) Suppose that R is simple. From Theorem 4 and Proposition 38, it follows that T is G -simple and that $Z(R)$ is a field (hence a domain). Now we show that L is a torsion group. Seeking a contradiction, suppose that there is $g \in L$ of infinite order and that there is an invertible $u \in T^G$ such that σ_g acts as conjugation by u . By Proposition 39, the element $x = 1 + u u_g$ belongs to $Z(R)$. Put $S = T \rtimes_{\sigma'}^\alpha \langle g \rangle$. By Lemma 43, we get that x is invertible in S . Suppose that $x^{-1} = \sum_{i=a}^b t_i u_{g^i}$, where $a, b \in \mathbb{Z}$, satisfy $a \leq b$ and that both t_a and t_b are non-zero. From the equality $x^{-1} x = 1$, we get, since g is of infinite order,

that either $t_a u_{g^a} = 0$ or $t_b \sigma_{g^b}(u) \alpha(g^b, g) u_{g^{b+1}} = 0$. This is clearly not possible. Thus, L is a torsion group.

Now suppose that T is G -simple, $Z(R)$ is a domain, and that L is a torsion group. From Theorem 4, it follows that R is simple if we can show that $Z(R)$ is a field. To this end, it is enough to show that each non-zero element of $Z(R)$ has a multiplicative inverse. From Lemma 45 it follows that $Z(R) = F \rtimes^\alpha L$. Since L is a torsion group it follows that every $x = f d_g$, $f \in F$, $g \in L$, satisfies $x^n \in F$ for some positive integer n . Since integral elements over an integral domain form a ring, it follows that any non-zero element $x \in Z(R)$ is integral over F . Let K denote the field of fractions of $Z(R)$. Then x^{-1} , considered as an element of K , belongs to $F[x] \subseteq Z(R)$.

(b) Suppose that R is simple. From (a) we get that T is G -simple and that L is a torsion group. From the proof of (a), we get that the field $Z(R)$ equals the group ring $F[L]$. Seeking a contradiction, suppose that there is some $g \in L \setminus \{e\}$. Since L is torsion, there is an integer $n \geq 2$ such that $g^n = 1$. But then $(1 - g)(1 + g + \cdots + g^{n-1}) = 1 - g^n = 0$. Therefore, $1 - g$ can not be invertible. Hence $L = \{e\}$.

Now suppose that T is G -simple and that $L = \{e\}$. From the proof of (a) we get that $Z(R) = F$ which is a field and, hence, a domain. Thus, simplicity of R follows from (a). \square

Definition 47. Suppose that $T \rtimes_\sigma G$ is a non-associative skew group ring. If G is a torsion-free finitely generated abelian group, with generators g_1, \dots, g_n , for some positive integer n , then $T \rtimes_\sigma G$ equals the *non-associative skew Laurent polynomial ring* $T[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}; \sigma_1, \dots, \sigma_n]$, where $\sigma_i := \sigma_{g_i}$, for $i \in \{1, \dots, n\}$. In that case, T is called $(\sigma_1, \dots, \sigma_n)$ -*simple* if there is no ideal I of T , other than $\{0\}$ and T , for which $\sigma_i(I) \subseteq I$ holds for all $i \in \{1, \dots, n\}$.

Theorem 48. *A non-associative skew Laurent polynomial ring*

$$T[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}; \sigma_1, \dots, \sigma_n]$$

is simple if and only if T is $(\sigma_1, \dots, \sigma_n)$ -simple and there do not exist $u \in \cap_{i=1}^n (T^\times)^{\sigma_i}$ and a non-zero $(m_1, \dots, m_n) \in \mathbb{Z}^n$ such that for every $t \in T$, the relation $(\sigma_1^{m_1} \circ \cdots \circ \sigma_n^{m_n})(t) = utu^{-1}$ holds.

Proof. We notice that \mathbb{Z}^n -simplicity of T is equivalent to $(\sigma_1, \dots, \sigma_n)$ -simplicity of T . The proof now follows from Theorem 42. \square

6. APPLICATION: CAYLEY-DICKSON DOUBLINGS

In this section, we apply Theorem 4 to obtain a simplicity result for Cayley-Dickson doublings (see Theorem 52) originally obtained by K. McCrimmon [9, Theorem 6.8(vi)] by other means. To do this we first need to translate what graded simplicity means for Cayley-Dickson doublings (see Proposition 49). After that, we describe the center of Cayley-Dickson doublings (see Proposition 51).

Throughout this section, A denotes a non-associative unital algebra with involution $*$: $A \rightarrow A$ defined over an arbitrary unital, commutative and associative ring of scalars K . Take a cancellable scalar $\mu \in K$. By this we mean that whenever $\mu k = 0$ (or $\mu a = 0$),

for some $k \in K$ (or some $a \in A$), then $k = 0$ (or $a = 0$). We can now construct a new algebra, the so-called *Cayley Double of A* , denoted by $C(A, \mu)$, as $A \oplus A$ with involution $(a, b)^* = (a^*, -b)$ and product $(a, b)(c, d) = (ac + \mu d^* b, da + bc^*)$. We can write this formally as $C(A, \mu) = A \oplus Al$ with product $(a + bl)(c + dl) = (ac + \mu d^* b) + (da + bc^*)l$ and involution $(a + bl)^* = a^* - bl$. An ideal I of A is called **-invariant* if $I^* \subseteq I$. The ring A is called **-simple* if there are no other *-invariant ideals of A other than $\{0\}$ and A itself. It is clear that if we put $R_0 = A$ and $R_1 = Al$, then this defines a \mathbb{Z}_2 -gradation on $R = C(A, \mu)$.

Proposition 49. *The ring $C(A, \mu)$ is graded simple if and only if A is *-simple and $\mu \in Z(A)^\times$.*

Proof. We first show the "only if" statement. Suppose that $C(A, \mu)$ is graded simple. Let I be a non-zero *-invariant ideal of A . Then $\langle I \rangle = I + Il$ is a non-zero graded ideal of $C(A, \mu)$. By graded simplicity of $C(A, \mu)$ we get that $\langle I \rangle = C(A, \mu)$. In particular, we get that $1 \in \langle I \rangle$. This implies that $1 \in \langle I \rangle \cap A = I$. Hence, $I = A$. This shows that A is *-simple.

Using that $\mu \in Z(A)$ and $\mu = \mu^*$ we get that $A\mu$ is a non-zero *-invariant ideal of A . Hence, $A\mu = A$, and in particular there is some $s \in A$ such that $s\mu = \mu s = 1$. Using Proposition 6 we conclude that $\mu \in Z(A)^\times$.

Now we show the "if" statement. Suppose that A is *-simple and that $\mu \in Z(A)^\times$. Let J be a non-zero graded ideal of $C(A, \mu)$. Put $I = J \cap A$. Using that μ is cancellable, we get that I is a non-zero ideal of A . The multiplication rule in $C(A, \mu)$ yields $ldl = \mu d^*$, for any $d \in A$. Using that $\mu \in Z(A)^\times$ we get $I^* = \mu^{-1}lIl \subseteq I$, and hence I is *-invariant. From *-simplicity of A we get that $I = A$. Therefore $A \subseteq J$ and hence $J = C(A, \mu)$. This shows that $C(A, \mu)$ is graded simple. \square

Proposition 50. *The center of $C(A, \mu)$ equals $Z_*(A) + Z_{**}(A)l$ where $Z_*(A) = \{a \in Z(A) \mid a = a^*\}$ and $Z_{**}(A) = \{a \in Z_*(A) \mid ab = ab^*, \forall b \in A\}$.*

Proof. This is [9, Theorem 6.8(xii)]. \square

Proposition 51. *The ring $Z(C(A, \mu))$ is a field if and only if either (i) $* = \text{id}_A$, $Z(A)$ is a field and μ is not a square in $Z(A)$, or (ii) $* \neq \text{id}_A$ and $Z_*(A)$ is a field.*

Proof. From Proposition 50, we get that $Z(C(A, \mu)) = Z_*(A) + Z_{**}(A)l$.

First we show the "only if" statement. Suppose that $Z(C(A, \mu))$ is a field. It is clear that $Z_*(A)$ is a field. Since $Z_{**}(A)$ is an ideal of $Z_*(A)$ we get two cases. Case 1: $Z_{**}(A) = Z_*(A)$. Since $1 \in Z_*(A)$, we get that $* = \text{id}_A$ and hence $Z(A) = Z_*(A) = Z_{**}(A)$ is a field. Therefore, $Z(C(A, \mu)) = Z(A) + Z(A)l = Z(A)[X]/(X^2 - \mu)$. If μ is a square in $Z(A)$, then $Z(C(A, \mu)) = Z(A) \times Z(A)$ or $l^2 = 0$, depending on the characteristic of $Z(A)$. In both cases $Z(C(A, \mu))$ is not a field. Hence we get that μ is not a square in $Z(A)$. Thus, (i) holds. Case 2: $Z_{**}(A) = \{0\}$. In this case $Z(C(A, \mu)) = Z_*(A)$ is a field. Seeking a contradiction, suppose that $* = \text{id}_A$. Then $ab = ab^*$, for all $b \in A$, and hence $Z_*(A) = Z_{**}(A) = \{0\}$ which is a contradiction. Thus, $* \neq \text{id}_A$. This shows that (ii) holds.

Now we show the "if" statement. Suppose first that (i) holds. Then $Z_{**}(A) = Z_*(A) = Z(A)$ and thus $Z(C(A, \mu)) = Z(A) + Z(A)l = Z(A)[X]/(X^2 - \mu)$, which is a field, since

μ is not a square in $Z(A)$. Now suppose that (ii) holds. Since $Z_{**}(A)$ is an ideal of the field $Z_*(A)$ we get that either $Z_{**}(A) = Z_*(A)$ or $Z_{**}(A) = \{0\}$. If $Z_{**}(A) = Z_*(A)$, then $*$ = id_A which is a contradiction. Therefore, $Z_{**}(A) = \{0\}$ and hence $Z(C(A, \mu)) = Z_*(A)$ is a field. \square

Theorem 52. *The ring $C(A, \mu)$ is simple if and only if A is $*$ -simple and either (i) A has trivial involution, $Z(A)$ is a field and μ is not a square in $Z(A)$, or (ii) A has non-trivial involution and $Z_*(A)$ is a field.*

Proof. The "only if" statement follows immediately from Proposition 49, Proposition 51 and Theorem 4. The "if" statement follows in the same way, by first observing that $\mu \in Z_*(A) \subseteq Z(A)$, and that μ therefore (in either case) is a non-zero element of a field, and thus invertible. \square

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